

On the Bethe Ansatz for Random Directed Polymers

J. P. Bouchaud¹ and H. Orland²

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We show that the inclusion of the (gapless) center-of-mass motion together with a functional integral representation of the Bethe wave function allows one to predict exactly the critical exponents for random directed polymers in $(1+1)$ dimensions. The corresponding amplitudes are computed; they compare satisfactorily with existing numerical data. Within a replica-symmetric theory, we find that the Green function of the polymer has the form recently proposed by Parisi.

KEY WORDS: Directed polymers; replica symmetry breaking; interface in random environments.

Directed polymers in random media have recently drawn considerable attention,⁽¹⁻⁷⁾ both because the equations describing their properties bear much resemblance with those appearing in a variety of other problems,^(1,3) and also because they may serve as a testing field for ideas and methods devised for the theory of spin-glasses.

One consequence of the presence of disorder is that the typical conformation of the polymer is "stretched" in the transverse direction, so that, far away, attractive impurities may be reached. "Anomalous" (e.g., non-Brownian) transverse fluctuations are thus expected, at least in sufficiently low dimensions.

Many results are now available, among which one may cite the exact solution on the Bethe lattice,⁽⁴⁾ $1/d$ expansions,⁽⁵⁾ and numerical simulations.^(6,7) Some analytical results have also been obtained in two dimen-

¹ Laboratoire de Physique Statistique de l'Ecole Normale Supérieure, associé au CNRS et à l'Université de Paris VI, 75231 Paris Cedex 05, France.

² Service de Physique Théorique (Laboratoire de l'Institut de Recherche Fondamentale du Commissariat à l'énergie atomique), Orme des Merisiers, 91191 Gif sur Yvette Cedex, France.

sions. In particular, the transverse extension is believed to scale as $X \approx L^{2/3}$, where L is the total length of the walk.

The replica method, followed by Kardar,⁽²⁾ leads to the study of an assembly of n one-dimensional quantum particles interacting via a potential given by the correlation function of the disorder. If this correlation function is a Dirac function, the ground state may be characterized exactly. This yields, for large L , the integer moments of the partition function⁽²⁾:

$$\langle Z^n \rangle = \exp\{-fL(n - n^3)\} \quad (1)$$

Then, following Zhang,⁽⁸⁾ one may argue that a probability distribution $P(F)$ for the free energy $F = -\text{Ln } Z$ which has *exponential* moments $\langle e^{nF} \rangle_P$ given by (1) can be explicitly constructed as

$$P(F) = \exp\{-a |F - F_0|^{3/2} L^{-1/2}\} \quad (2)$$

with $F_0 = fL$. From (2) one concludes that the typical fluctuations of F are of order $\Delta F \approx L^\omega$ with $\omega = 1/3$. Then, using the scaling relation⁽¹⁾ $\Delta F \approx X^2/L$, one finally obtains $X \approx L^\nu$ with $\nu = 2/3$.

This line of reasoning is not entirely satisfying, for the following reasons:

(a) Since the moments $\langle Z^n \rangle$ grow as $\exp n^3 \gg n^n$, the probability distribution of Z (and hence the moments of $\langle \text{Ln } Z \rangle$) is not uniquely determined. In other words, (2) may only give information on large deviations (i.e., the tails of the distributions) and not on the typical fluctuations.

(b) The scaling relation⁽¹⁾ $\omega = 2\nu - 1$ must be used to obtain the behavior of X , which one would like to infer directly from the replica calculation. This would also allow one, in principle, to obtain the prefactor C defined as

$$C = \lim_{L \rightarrow \infty} \langle X^2 \rangle / L^{4/3}$$

(c) Finally, a "ground-state dominance" hypothesis has been made⁽²⁾ to obtain (1). In other words, (1) is valid only if there is a *gap* in the Hamiltonian spectrum describing the n fictitious particles alluded to above. This hypothesis is, however, surely not justified, since the kinetic energy of the center of mass can be made arbitrarily small. The aim of this note is to show that if one includes the whole band of center-of-mass plane waves, the result $X \approx L^{2/3}$ naturally follows. We also discuss how one may obtain the prefactor C , and also the nonextensive corrections to the average free energy (of order $L^{1/3}$).

We thus start from the continuum path integral formulation of Z :

$$Z = \int \mathcal{D}x(t) \exp - \int dt \{ \dot{x}^2/4D - \beta V(x(t), t) \} \tag{3a}$$

with $\langle V(x, t) V(x', t') \rangle = V^2 \delta(x - x') \delta(t - t')$. The initial point is fixed at $x(0) = 0$, whereas the final point is free.

Hence

$$\langle Z^n \rangle = e^{n(\beta V^2)L/2a} \int \mathcal{D}x_\alpha(t) \exp - \int dt \left\{ \sum_\alpha \dot{x}_\alpha^2/4D - \beta^2 V^2 \sum_{\beta > \alpha} \delta(x_\alpha(t) - x_\beta(t)) \right\} \tag{3b}$$

where a is a short-distance cutoff needed to define the model properly. Now defining the Hamiltonian

$$\mathcal{H}_n = -D \sum_\alpha \partial^2/\partial x_\alpha^2 - \beta^2 V^2 \sum_{\beta > \alpha} \delta(x_\alpha - x_\beta) \tag{4}$$

one may express $\langle Z^n \rangle$ as

$$\langle Z^n \rangle = \int dx_1 \cdots dx_n \langle x_1 \cdots x_n | \exp - L\mathcal{H}_n | 0 \cdots 0 \rangle \tag{5}$$

Only the low-energy states of \mathcal{H}_n will matter for L large. The ground state of \mathcal{H}_n is easily shown to be⁽²⁾

$$|\Psi_0\rangle = \mathcal{N}^{-1} \exp \left(-\lambda \sum_{\beta > \alpha} |x_\alpha - x_\beta| \right) \tag{6}$$

with $4\lambda D = \beta^2 V^2$. The ground-state energy is then $E_0 = -\lambda^2 D n(n^2 - 1)/3$. The low-lying excitations are simply constructed by allowing the center of mass to have a nonzero momentum:

$$|\Psi_k\rangle = \mathcal{N}^{-1} \exp \left(ik \sum_\alpha x_\alpha \right) \exp \left(-\lambda \sum_{\beta > \alpha} |x_\alpha - x_\beta| \right) \tag{7}$$

with the associated energy $E_k = E_0 + nDk^2$. For large L , we shall only keep the subspace spanned by the $|\Psi_k\rangle$ and write

$$\exp(-L\mathcal{H}_n) = \exp(-LE_0) \int dk (nLD/\pi)^{1/2} \exp(-nLDk^2) |\Psi_k\rangle \langle \Psi_k| \tag{8}$$

Note that in the case $V^2 = 0$, the center-of-mass motion is indeed necessary

to recover the “free” diffusion propagator. Now, following Parisi,⁽⁹⁾ we will use a Gaussian representation for $\exp(-\lambda \sum_{\beta > \alpha} |x_\alpha - x_\beta|)$:

$$\exp\left\{-\lambda \sum_{\beta > \alpha} |x_\alpha - x_\beta|\right\} \equiv \int \mathcal{D}\varphi \exp\left[-(1/4\lambda) \int dx (\partial\varphi/\partial x)^2 + \sum_x \varphi(x_\alpha)\right] \quad (9)$$

with the normalization

$$\int \mathcal{D}\varphi \exp\left[-(1/4\lambda) \int dx (\partial\varphi/\partial x)^2\right] \equiv 1$$

This representation allows us to show that $\Psi_0(x) \equiv 1$ and $\mathcal{N} = 1$ for $n = 0$.

From (5) and (9) one then obtains

$$\begin{aligned} \langle Z^n \rangle &= \exp(-LE_0) \int \mathcal{N}^{-2} dk (nLD/\pi)^{1/2} \exp(-nLDk^2) \\ &\quad \times \int \mathcal{D}\varphi \exp\left[-(1/4\lambda) \int dx (\partial\varphi/\partial x)^2\right] \\ &\quad \times \int \pi dx_x \exp\left[ik \sum_x x_\alpha + \sum_x \varphi(x_\alpha)\right] \end{aligned} \quad (10a)$$

For integer n , φ and x play symmetric roles: they are both dynamic fields. In the limit $n \rightarrow 0$, however, φ becomes a “quenched” field, since one has to compute expression of the form

$$\begin{aligned} &\int \mathcal{D}\varphi \exp\left[-(1/4\lambda) \int dx (\partial\varphi/\partial x)^2\right] \\ &\quad \times \left\{ \int dx O_\varphi(x) \exp[ikx + \varphi(x)] \middle/ \int dx \exp[ikx + \varphi(x)] \right\} \end{aligned} \quad (10b)$$

Hence, in the limit $n = 0$, we maintain that $\langle Z^n \rangle$ should be computed as the *quenched* average over the φ field of

$$\exp(-LE_0) \int \pi dx_x \exp\left[-(1/4nDL) \left(\sum_x x_\alpha\right)^2 + \sum_x \varphi(x_\alpha)\right] \quad (11)$$

(the integral over k has been performed): using the φ fields naturally defines a procedure (taking φ as a quenched field) which is difficult to transpose directly to the Bethe wave function (6).³ One now rescales x and φ as

$$x = L^{2/3}y \quad \text{and} \quad \varphi(x) = L^{1/3}\psi(y) \quad (12)$$

³ The direct computation of $\langle X^2(L) \rangle$ using (8) has been performed by Mézard.⁽⁷⁾ It yields

$$\langle X^2(L) \rangle \approx L/n - 1/n^4$$

which shows that the limit $n \rightarrow 0$ cannot be reached directly with expression (8).

This rescaling leaves invariant the ψ measure. Hence,

$$\int \pi dx_\alpha \exp - \left[(1/4nDL) \left(\sum_\alpha x_\alpha \right)^2 + \sum_\alpha \varphi(x_\alpha) \right] \\ = L^{2n/3} \int \pi dy_\alpha \exp \left\{ -L^{1/3} \underbrace{\left[\left(\sum_\alpha y_\alpha \right)^2 / 4nD + \sum_\alpha \psi(y_\alpha) \right]}_{\mathcal{A}} \right\} \quad (13)$$

For fixed n and large L ,⁴ we search the saddle point of the expression in braces \mathcal{A} for a fixed ψ . Writing $\rho(y) = n^{-1} \sum_\alpha \delta(y - y_\alpha)$ and $Y = n^{-1} \sum_\alpha y_\alpha$, the functional minimization of \mathcal{A} with respect to $\rho(y)$ leads to

$$yY = 2D\psi(y) \quad (14)$$

which cannot be satisfied, for a typical ψ , for all y . Only a finite set of points $\{y_i\}$ are solutions of (14), and \mathcal{A} then reads

$$\mathcal{A} = -n/4DY^2$$

In order to minimize \mathcal{A} , we assume that all y_α must be equal (“replica-symmetric solution”) to a value ξ which minimizes the one-body energy:

$$E(\psi) = \xi^2/4D + \psi(\xi) \quad (15)$$

Physically, this represents the energy of a point particle attached to the origin with a spring, and subject to a potential ψ the slope of which is a Gaussian random variable of variance 2λ . This model was studied in refs. 10 and 11. In the limit $nL^{1/3}$ large and assuming “replica symmetry” [which amounts to discarding situations where quasidegenerate minima of (15) exist], one obtains $\langle Z^n \rangle$ as

$$\int \mathcal{D}\psi \exp \left[-(1/4\lambda) \int dx (\partial\psi/\partial x)^2 \right] \exp[-nL^{1/3} E_{\min}(\psi)] \quad (16)$$

which has the following interpretation: for a given L , one may associate to each sample a configuration $\{\varphi(x)\}$ chosen with the weight (9). Then the “toy-model” partition function $z = \int d\xi \exp[-L^{1/3} E(\psi(\xi))]$ is computed for each sample. If z is correctly estimated as $\exp[-L^{1/3} E_{\min}(\psi)]$, then (16) simply expresses that $\langle Z^n \rangle$ is the average over φ of z^n , and hence that

⁴ As usual in disordered systems (see, e.g., ref. 13), L must go to infinity before the limit $n=0$ is taken.

the toy model (15) contains the full physics of the original problem.⁵ Thus φ , which initially appeared through a mathematical “trick” [Eq. (9)] has (in this approximation scheme) a direct physical interpretation. As emphasized in ref. 9, this is best seen on the Burgers equation approach to the problem: to each sample of a given length L , one may indeed associate an effective potential φ acting only on the “head” of the polymer. The “miracle” comes from the fact that whereas the true potentials V have short-range correlations, the effective potential φ has long-range correlations $\langle [\varphi(X) - \varphi(Y)]^2 \rangle \approx |X - Y|$.

A lot of information can be deduced from this simplified model. For example, the moments of the transverse extension and the energy may be obtained numerically (similar calculations were performed in ref. 11) $\langle \dots \rangle$ now means average over ψ :

$$\begin{aligned} \langle |\xi| \rangle &= 0.82(D\beta^2 V^2)^{1/3} \\ \langle \xi^2 \rangle^{1/2} &= 1.02(D\beta^2 V^2)^{1/3} \\ \langle \xi^4 \rangle^{1/4} &= 2.35(D\beta^2 V^2)^{1/3} \\ \langle E \rangle &= -0.76(\beta^4 V^4/D)^{1/3} \end{aligned} \quad (17)$$

To obtain these numbers, we generated the Gaussian process ψ . We then searched for the value of ξ which minimizes the energy (15). We then averaged over 20,000 configurations of ψ to obtain (17). The results yield, for the original problem [through the scalings (12)],

$$F = -\langle \ln Z \rangle = +L\lambda^2 D/3 + L^{1/3} \langle E \rangle \quad (18)$$

Note that the $L^{1/3}$ term appears as a *correction* in the average free energy. One also obtains $C = 1.02(D\beta^2 V^2)^{1/3}$. This compares very well with numerical results⁶ of Kardar⁽¹⁾ ($\beta^2 V^2 = 5/6$ and $D \approx 0.33$ or ≈ 0.03), which

⁵ That this toy model retains much of the original problem has also been suggested by Parisi.⁽⁹⁾ However, instead of introducing the center-of-mass motion as in Eq. (8), he constructs heuristically the Green function of the problem by multiplying the Bethe wave function (6) by the product of the free Green functions of the individual particles:

$$G_{\text{Parisi}}(x_1, \dots, x_n) = \mathcal{N}^{-1} \exp\left(-\sum_{\alpha} x_{\alpha}^2/4DL\right) \exp\left(-\lambda \sum_{\beta > \alpha} |x_{\alpha} - x_{\beta}|\right)$$

which also gives (20), (21) in the limit $n = 0$.

⁶ Disorder must be small if one wants to discretize faithfully Eq. (3), i.e., in such a way that the microscopic diffusion constant D and disorder variance $(\beta V)^2$ really determine (3). For strong disorder [$(\beta V)^2 \gg 4D$], the very first steps of the walk are already sensitive to the presence of impurities: this “renormalizes” D and $(\beta V)^2$. Then (3) must be understood in a

give $C \approx 0.98$. One may also obtain the free energy fluctuations: the term of order n^2 in the expansion of $\langle Z^n \rangle$ is given by

$$\langle \Delta F^2 \rangle = L^{2/3} (\langle E^2(\varphi) \rangle - \langle E(\varphi) \rangle^2) \tag{19}$$

Hence $\langle \Delta F^2 \rangle^{1/2} \approx 0.42 L^{1/3} (\beta^4 V^4 / D)^{1/3}$. It would be interesting to compare the prefactors of higher moments of X . One may also obtain the probability distribution to find the “head” of the polymer at site X after “time” L : using the properties of $\varphi(X)$, one finds

$$\langle \ln P(X, L) \rangle = -X^2 / 4DL \tag{20}$$

and

$$\langle (\ln P(X, L))^2 \rangle - \langle \ln P(X, L) \rangle^2 = 2\lambda |X| \tag{21}$$

which coincide with the results recently proposed by Parisi (see footnote 5).

The behavior of $\langle P(X, L) \rangle$ for large X can also be computed: one finds⁽¹¹⁾ that $\ln \langle P(X, L) \rangle \approx (X/L^{2/3})^3$ for $X \gg L^{2/3}$. The asymptotic shape of $P(\ln Z)$ in this toy model can be characterized⁽¹²⁾: $P(\ln Z) \approx \exp(-(\ln Z)^{3/2})$, in agreement with Eq. (2).⁽⁸⁾

These predictions have been checked numerically by Mézard⁽¹⁾: while (20) is well reproduced, (21) seems to be only valid for values of $y = X/L^{2/3}$ not too large ($y \leq 2$): this again suggests that the existence of quasidegenerate solution minima of (15) might lead to a breaking of the “replica symmetry” in (13), i.e., that one should put some fraction fn of the y_x in one well and the other $(1-f)n$ in the other quasidegenerate wells (see note added in proof). In this case, the simple picture based on the toy model (15) cannot be correct [see Eq. (16) and the discussion that follows]. Replica symmetry breaking, related to the existence of several pure states with free energy difference $O(L^{1/3})$, has been suggested independently by Parisi⁽⁹⁾ and by Mézard.⁽⁷⁾ It is argued in ref. 9 that low-energy states of \mathcal{H}_n (for $n \rightarrow 0$) can be constructed by forming well-separated packets (for example, two clusters containing each $n/2$ particles). Inside each cluster, one chooses the wavefunction (6): the energy of the obtained state is then argued to be very close to the true ground state when $n \rightarrow 0$. It is reasonable to think that the existence of several

coarse-grained sense. The comparison with the numerical data of ref. 1 is meaningful since a change of a factor ≈ 10 in D does not change the obtained value of the prefactor C , showing that one is indeed in a weak-disorder regime. On the contrary, no quantitative comparison can be made with the data of ref. 7, since in that case, disorder is strong [$D = 1/2, (\beta V)^2 = 49$].

quasidegenerate ground states of (15) could be the correct coding of this effect. This replica symmetry breaking is expected to affect the numerical values given in Eq. (17), although the lowest moments of ξ should not be dramatically shifted.

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NOTE ADDED IN PROOF

In a recent paper, Fisher and Huse (Preprint July 1990) have argued that the correlation function of the effective potential $\varphi(X)$ should saturate for distances $\approx L^{2/3}$ since two walks starting from two points $L^{2/3}$ apart cannot “interfere” with each other. A (phenomenological) way to obtain this saturation is to add a “mass term” $m_0^2 \varphi^2$ to the ill-defined (at $k=0$) “action” $(1/4\lambda) \int dx (\partial\varphi/\partial x)^2$. This leads to $\langle \psi(X)\psi(Y) \rangle = (1/m_0) [1 - \exp(-m_0 |x-y|)]$, which compares satisfactorily with Mezard’s data.⁽⁷⁾

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